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FUZZY NODE CONNECTIVITY OF SOME CLASSES OF GRAPHS

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Abstract. The parameters defining vulnerability in graph theory can be used as an indicator of the service quality over the network, if nodes or links fail. In the field of graph theory, numerous vulnerability parameters have been defined, including toughness, rupture degree, tenacity, integrity, connectivity, and others. Fuzzy graphs, a specific type of graphs, provide a more effective method of modelling real-world problems than other graphs. This is due to the fact that the uncertainties inherent in the problems can be expressed in a more realistic manner through the use of membership values. However, despite this advantage, there has been limited research conducted on the vulnerability parameters in fuzzy graphs. In this paper, the node connectivity parameter for fuzzy wheel graphs, fuzzy cycle graphs, and fuzzy star graphs are researched and some general formulas are obtained. Additionally, the algorithms are given to find the strength of connectedness between any two nodes and the node connectivity of the given fuzzy graph.

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1 Introduction

Networks are composed of nodes and links that connect them. Examples of such networks include biological networks, social networks, communication networks, and so on. One of the mathematical tools that can be used to model networks is graph theory. Graph theory provides a variety of analytical tools for networks, including graphs defined in different ways and parameters defined for different graph types. One such parameter is the vulnerability parameter (Barefoot & Entringer, 1978).

A network's vulnerability can be measured using graph parameters like connectivity (Harary, 1969), integrity (Barefoot & Entringer, 1978), tenacity (Cozzens et al., 1995), toughness (Chvatal, 1973), and rupture degree (Li et al., 2005). Connectivity is the number of non-functioning elements in a graph which is the oldest parameter studied.

A specific type of graphs is called a fuzzy graph. Fuzzy graphs allow any real number between 0 and 1 to represent the degree of flow or membership value that indicates how related any two nodes are. Modeling by grading the relations will provide a more accurate depiction of reality in a world where things are not always black and white. Fuzzy graphs are especially useful when there is uncertainty about the nodes and/or links (arcs) in question. Fuzzy graphs are therefore a useful tool for more accurately simulating real-world issues. Rosenfeld (1975) and Yeh & Bang

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(1975) independently defined fuzzy graphs in 1975 based on the ideas of fuzzy sets and fuzzy relations established in Zadeh (1965).

While the current literature on other types of graphs gives a thorough definition of vulnerability parameters, they are less commonly specified in the context of fuzzy graphs. Rosenfeld (Rosenfeld, 1975) discovered fuzzy analogues of numerous core graph-theoretical structural concepts and conceptions of connectedness, whereas (Yeh & Bang, 1975) proposed a framework for comprehending fuzzy graph connectivity.

The literature defines a number of vulnerability parameters for fuzzy graphs. For example, (Mathew & Sunitha, 2010) offered a revision of the connectivity parameter based on Rosenfeld's notion of connectedness Rosenfeld (1975), Ali et al. (2018) defined average fuzzy vertex connectivity and total fuzzy vertex connectivity Additionally, Saravanan et al. (2015) established an integrity parameter for fuzzy graphs, and Binu et al. (2019) defined a connectivity index.

In this study, the node connectivity parameter is studied for some fuzzy graphs, such as fuzzy wheel graphs, fuzzy cycle graphs, and fuzzy star graphs. The general formulas are derived based on the membership values. Furthermore, the algorithms are designed to ascertain the strength of connectedness between any two nodes and the node connectivity of the provided fuzzy network.

2 Preliminaries

In this section some basic definitions are given. The definitions not given here can be found in Altundag (2021); Mathew & Sunitha (2010); Mordeson & Nair (2000).

A fuzzy graph $G:(V,\sigma,\mu)$ is a nonempty set V together with a pair of functions $\sigma: V \to [0,1]$ and $\mu: V\times V \to [0,1]$ such that for all $u,v \in V$, $\mu(u,v) \leq \sigma(u)\wedge\sigma(v)$ (Mordeson & Nair, 2000).

An edge of a fuzzy graph is called arc. If an arc has the least membership value in a fuzzy graph, then it is called weakest arc. $d(\mu) = \wedge\{\mu(u,v) \mid (u,v) \in \mu^*\}$ is called the depth of μ and $h(\mu) = \vee\{\mu(u,v) \mid (u,v) \in \mu^*\}$ is called the height of μ (Bhutani et al., 2004).

Multimin cycle refers to a fuzzy cycle that includes more than one weakest arc.It is said to be a locamin cycle if each node is connected to a weakest arc of the cycle (Bhutani & Rosenfeld, 2003b).

A multimin cycle is a fuzzy cycle with more than one weakest arc and a locamin cycle is a fuzzy cycle with each node incident with a weakest arc (Bhutani & Rosenfeld, 2003b).

The strength of connectedness between two nodes, designated as u and v, is defined as the maximum value of the strengths observed across all paths between the two nodes and is denoted by $CONN_G(u,v)$ (Rosenfeld, 1975).

A fuzzy graph $G:(V,\sigma,\mu)$ is connected if $CONN_G(u,v) > 0$ for every u,v in σ^*. It is acknowledged that throughout this paper, G is assumed to be connected.

If (u,v) is an arc of G and $\mu(u,v) > CONN_{G-(u,v)}(u,v)$, then (u,v) is called α-strong arc, if $\mu(u,v) = CONN_{G-(u,v)}(u,v)$, then β-strong and if $\mu(u,v) < CONN_{G-(u,v)}(u,v)$ a δ-arc (Mathew & Sunitha, 2009). If all the arcs on a path are strong, then it is called a strong path (Bhutani & Rosenfeld, 2003b).

In the event that the removal of an arc results in a reduction of the strength of connectedness between any pair of nodes, the arc is called a fuzzy bridge (Mathew & Sunitha, 2009). Similarly, if the removal of a node diminishes the strength of connectedness between any pair of nodes, then that node is referred to as a fuzzy cut node (Mathew & Sunitha, 2009). A node is designated as a fuzzy end node of G if it is connected to at most one strong arc (Bhutani & Rosenfeld, 2003a).

The sequence of arc strengths, denoted by $\{q_1, q_2, ...q_m\}$, is a set of numbers that represents the strength of each arcs of the fuzzy graph $G:(V,\sigma,\mu)$ where $q_1 \leq q_2 \leq q_2 \leq ... \leq q_{m-1} \leq q_m$ (Altundag, 2021).

3 Fuzzy Node Connectivity

Vertex connectivity of fuzzy graphs was first defined in 1975 by Yeh and Bang as the minimum weight of a disconnection of a fuzzy graph (Yeh & Bang, 1975).

Definition 1. *(Yeh & Bang, 1975) In the context of fuzzy graphs, a disconnection refers to a vertex set, denoted by D, which, if removed, results in a disconnected or a single vertex graph. The weight of D is defined to be $\Sigma_{v\in D}\{\min \mu(v,u) \mid \mu(v,u) \neq 0\}$.*

Definition 2. *(Yeh & Bang, 1975) The vertex connectivity of a fuzzy graph is defined as the minimum weight of a disconnection in G.*

This definition depends on the disconnection of the fuzzy graphs and so this parameter is more related to graphs than fuzzy graphs. In 2010, Mathew and Sunitha redefined the concept of connectivity called fuzzy node connectivity regarding the strength of connectedness other than disconnection (Mathew & Sunitha, 2010).

Definition 3. *(Mathew & Sunitha, 2010) Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph. $S = \{v_1, v_2, ..., v_m\}$ is a fuzzy node cut if either, for some pair of nodes $u, v \in \sigma^*$ $CONN_{G-S}(u,v) < CONN_G(u,v)$ where $u, v \neq v_i$, $i = 1, 2, ..., m$ or $G - S$ is trivial.*

In Bhutani & Rosenfeld (2003b), it is shown that there exists at least one strong arc incident on every node of nontrivial connected fuzzy graph. Motivated by this following definitions were defined in (Mathew & Sunitha, 2010).

Definition 4. *(Mathew & Sunitha, 2010) The strong weight of a set S is the sum of the minimum of the weights of strong arcs incident on each node of S.*

Definition 5. *(Mathew & Sunitha, 2010) The fuzzy node connectivity of a connected fuzzy graph G is defined as the minimum strong weight of fuzzy node cuts of G, denoted by $\kappa(G)$.*

The fuzzy node connectivity parameter applied to some types of fuzzy graphs and general formulas are extracted.

Lemma 1. *Let $G : (V, \sigma, \mu)$ be a fuzzy cycle $\forall u \in V(G)$. Then*

$$s(u) \geq d(\mu).$$

Proof. In a fuzzy cycle graph, the arcs are either $\beta - strong$ or $\alpha - strong$ arcs (Mathew & Sunitha, 2009). Since the strength of a vertex is equal to the smallest value of the adjacent strong arcs, $\forall u \in V(G)$ it is obtained that

$$s(u) \geq d(\mu).$$

□

Theorem 1. *Let us consider a self-centered fuzzy cycle graph $G : (V, \sigma, \mu)$ of order n. We assume that for $i = 1, ..., n-1$, we have $e_i = (u_i, u_{i+1})$ and $e_n = (u_n, u_1)$. Let $0 < t < s \leq 1$.*
Case i)
***1)** If $\mu(e_i) = t$ or $\mu(e_i) = s$ where $i = 1, ..., n$;*
***2)** If $\mu(e_{2i-1}) = t$ and $\mu(e_{2i}) = s$ for $i = 1, 2, ..., \frac{n}{2}$ where n is an even integer;*
***3)** If $\mu(e_{2i-1}) = \mu(e_n) = t$ and $\mu(e_{2i}) = s$ for $i = 1, 2, ..., \frac{n-1}{2}$ where $n = 4k + 1$ and $k = 1, 2, ...,$ then*

$$\kappa(G) = 2d(\mu)$$

***Case ii)** If $\mu(e_{2i-1}) = \mu(e_n) = s$ and $\mu(e_{2i}) = t$ for $i = 1, 2, ..., \frac{n-1}{2}$ where $n = 4k - 1$ and $k = 1, 2, 3, ...,$ then*

$$\kappa(G) = \min\{h(\mu), 2d(\mu)\}.$$

Proof. **Case i)** In the case of a self-centered fuzzy cycle graph G satisfying one of the conditions outlined in 1, 2, and 3, it can be determined that G is a multimin cycle, as it possesses at least two weakest arcs. (Bhutani & Rosenfeld, 2003a). Moreover, it can be demonstrated that G is a locamin cycle, given that each node is incident to a weakest arc. It can be stated that a multimin cycle graph is a locamin cycle if and only if the cycle graph does not contain a fuzzy cut node. (Bhutani & Rosenfeld, 2003a). Hence, fuzzy cycle graph G does not have a fuzzy cut node.
Let S be a fuzzy node cut of G where $|S| \geq 2$. It can be easily seen that each path between any two nonadjacent nodes in a fuzzy cycle graph G is a strongest path since G is both a mutimin cycle and a locamin cycle. For this reason, $s(u_i) = d(\mu)$ for $i =, 2, ..., n$.
Since, the number of disjoint strongest paths between the nodes u and v in a fuzzy cycle graph is at most 2 In accordance with Menger's theorem, the cardinality of the minimal u-v reducing set is a minimum of 2. (Mathew & Sunitha, 2013).
Therefore, if $|S| = 2$, then we get $s(S) = |S|.d(\mu) \Rightarrow s(S) = 2.d(\mu)$. Hence;

$$\kappa(G) = 2d(\mu).$$

Case ii) Let G be a self-centered fuzzy cycle graph for $k = 1, 2, 3, ...$ with $n = 4k - 1$ nodes and let $\mu(e_{2i-1}) = \mu(e_n) = s$, and $\mu(e_{2i}) = t$ for $i = 1, 2, ..., \frac{n-1}{2}$. Fuzzy cycle graph G is a multimin cycle because it has more than one weakest arc. u_1 is the only node that is incident to two $\alpha - strong$ arcs where $\mu(e_1) = \mu(e_n) = h(\mu)$. So u_1 is a fuzzy cut node. Thus we get $s(u_1) = h(\mu)$ and $s(u_i) = d(\mu)$ for $i = 2, 3, ..., n$ $\forall(u_i) \in V(G)$.
Let S be a fuzzy node cut of a fuzzy cycle graph G.
a) If $|S| = 1$, then it is obtained that $s(S) = s(u_1) = h(\mu)$ since $S = \{u_1\}$. Therefore

$$\kappa(G) = h(\mu) \tag{1}$$

b) If $|S| = 2$, then for $u_1 \in S$ we have

$$s(S) = h(\mu) + d(\mu) \tag{2}$$

and for $u_1 \notin S$ we get

$$s(S) = 2d(\mu) \tag{3}$$

c) If $|S| \geq 3$, then we obtain

$$s(S) \geq |S|d(\mu) \geq 3d(\mu) \tag{4}$$

Hence, according to the definition by (1),(2),(3) and (4)

$$\kappa(G) = \min\{2d(\mu), h(\mu)\}.$$

□

Theorem 2. *If $G : (V, \sigma, \mu)$ is a regular fuzzy cycle graph then the fuzzy node connectivity value of G is*

$$\kappa(G) = 2d(\mu).$$

Proof. Let G be a regular fuzzy cycle graph. Then for all $e \in E(G)$ the values of $\mu(e)$ are either constant or alternate arcs have the same membership values. For this reason, there are two cases.
Case i) If all the values of $\mu(e)$ for all $e \in E(G)$ are fixed, then all arcs of the regular fuzzy cycle graph are weakest arc. Therefore each node in G is a central node and G corresponds to a self-centered fuzzy cycle graph (Tom & Sunitha, 2015). By Theorem (1),

$$\kappa(G) = 2d(\mu).$$

Case ii) In the assumption that the membership values of the alternate arcs are identical, then the regular fuzzy cycle graph is an even cycle (Gani & Radha, 2008). For this reason, fuzzy cycle graph G is a self-centered fuzzy cycle graph (Tom & Sunitha, 2015). By Theorem (1),

$$\kappa(G) = 2d(\mu).$$

□

Theorem 3. *Let us consider a fuzzy graph by $G : (V, \sigma, \mu)$ which contains a single weakest arc. If G^* is a cycle graph, then the fuzzy node connectivity value of G is*

$$\kappa(G) = q_2$$

Proof. Let the arc (u_1, u_2) be the weakest are where $u_1u_2...u_n$ represent the nodes of G. Thus the nodes designated as u_1 and u_2 are fuzzy end nodes, while the nodes labelled u_i for $i = 1, 2, 3, ..., n$ are fuzzy cut nodes (Bhutani et al., 2004). It can be demonstrated that the function $\mu(u_1, u_2) = d(\mu) = q_1$ is a δ-arc, while the remaining arcs are α-strong. Consequently, the values of $s(u_i)$ are equal to $= q_j$, where $i = 1, ..., n$ and $j = 2, ..., m$.
Case i) Let S be a fuzzy node cut of G. If $|S| = 1$, then for $u_i \in S$ where $i = 3, ..., n$.

$$\kappa(G) = \min\{q_j \mid j = 2, ..., m\} = q_2 \tag{5}$$

Case ii) If $|S| \geq 2$, then $s(S) = \sum_{i=1}^{n} s(u_i) \geq \sum_{i=2}^{m} q_j$. In accordance with the definition of fuzzy node connectivity, S should be chosen so that the strong weight of S is as small as possible. Thus, S can be chosen where $|S| = 2$ such that $s(S) = q_2 + q_3$. In this case, we obtain

$$\kappa(G) = q_2 + q_3 \tag{6}$$

By the equations (5) and (6),

$$\kappa(G) = q_2.$$

□

Theorem 4. *Let the fuzzy wheel graph W_n consist of nodes u_n as the fuzzy hub and nodes $u_1, u_2, ..., u_{n-1}$ on the fuzzy cycle C_{n-1}. It is assumed that $\mu(u_n, u_i) \leq \mu(e)$. In this case, the value of membership for the arcs situated on the cycle, denoted by $\mu(e)$, are distinct for all $e \in E(C_{n-1})$. It can be seen that the values of $\mu(u_n, u_i)$ are equal to the weakest arc on the cycle for $i = 1, ..., n-1$. Consequently, the fuzzy node connectivity value is*

$$\kappa(W_n) = \begin{cases} d(\mu), & \textit{if there is a FCN;} \\ 3d(\mu), & \textit{otherwise.} \end{cases}$$

Proof. On the fuzzy wheel graph W_n, every path is a strong path. Therefore each arc of W_n is either $\alpha - strong$ or $\beta - strong$. Thus the weakest arcs are $\beta - strong$. For $i = 1, ..., n-1$ all the nodes u_i are adjacent to the fuzzy hub. Then, $s(u_i) = d(\mu)$. Since $\mu(u_n, u_i) \leq \mu(e)$, based on the existence of the fuzzy cut node of W_n, there are two cases.

Case i) Let consider a fuzzy wheel graph W_n which contains a fuzzy cut node, represented by x and let $S = \{x\}$ be a fuzzy node cut of W_n. As the fuzzy hub is not a fuzzy cut node, fuzzy cycle C_{n-1} should contain the fuzzy cut node x. If and only if a node x on a fuzzy cycle is a common node of two fuzzy bridges, then it is a fuzzy cut node. If and only if an arc of a fuzzy cycle is a fuzzy bridge, then the arc is $\alpha - strong$ (Mathew & Sunitha, 2009).

For this reason, the incident arcs (u, x) and (x, v) on the fuzzy cycle at node x are α-strong. Removal of node x from the wheel results in

$$CONN_{G-S}(u, v) = d(\mu) < CONN_G(u, v) = \min\{(u, x), (x, v)\}.$$

Hence $s(S) = d(\mu)$ since $s(u_i) = d(\mu)$ for $i = 1, ..., n$. For every fuzzy cut node S, we have $s(S) \geq |S|d(\mu)$. Thus

$$\kappa(G) = \min\{s(S)\} = d(\mu)$$

Case ii) Let consider a fuzzy wheel graph W_n without a fuzzy cut node. For any non-strong arc (u, v) of the wheel W_n, every path between nodes u and v is the strongest path because every path between u and v has the same strength. There are at most three internally disjoint, strongest paths between u and v. According to Menger's Theorem, the minimal $u - v$ strength reducing set has a cardinality of at most three (Mathew & Sunitha, 2013). As a result, we get $s(S) = 3d(\mu)$ since $|S| = |S_G(u, v)| = 3$ and

$$\kappa(G) = \min\{s(S)\} = 3d(\mu).$$

□

Theorem 5. *Let the fuzzy wheel graph W_n consist of nodes u_n as the fuzzy hub and nodes $u_1, u_2, ..., u_{n-1}$ on the fuzzy cycle C_{n-1}. It is assumed that $\mu(u_n, u_i) \geq \mu(e)$. In this case, the memberships values associated with the arcs within the cycle, denoted by $\mu(e)$, are distinct for all $e \in E(C_{n-1})$ and for all values of i between 1 and $n - 1$, the value of $\mu(u_n, u_i)$ is constant. Thus, the fuzzy node connectivity value is*

$$\kappa(W_n) = h(\mu).$$

Proof. **Case i)** Let assume the value of $\mu(u_n, u_i)$ be constant for $i = 1, ..., n-1$ and $\mu(u_n, u_i) > \mu(e)$ for $e \in E(C_{n-1})$. The fuzzy wheel graph W_n has at most one strong arc incident to the nodes u_i for $i = 1, ..., n - 1$ lying on the cycle. Therefore, each node, u_i, can be defined as a fuzzy end node. In order for a node to be defined as a fuzzy cut node, it must be incident to at least two strong arcs. As a result, the nodes u_i are not classified as fuzzy cut nodes (Bhutani & Rosenfeld, 2003a). However, since $\mu(u_n, u_i) > \mu(e)$, the node u_n is a cut node and for $i = 1, ..., n$. we get $s(u_i) = h(\mu)$.
Assume that S denote a fuzzy node cut of W_n.
For $|S| = 1$, the only set is $S = \{u_n\}$. Thus $s(S) = h(\mu)$.
For $|S| \geq 2$, $s(S) = |S|.h(\mu) \geq 2h(\mu)$.
Therefore by the definition

$$\kappa(G) = h(\mu).$$

Case ii) Let assume the value of $\mu(u_n, u_i)$ be constant for $i = 1, ..., n-1$ and $\mu(u_n, u_i) \geq \mu(e)$ for $e \in E(C_{n-1})$. Let u_n be the fuzzy hub. For all $u, v \in V(C_{n-1})$, the $u-v$ paths of length two are strongest paths. So for $i = 1, ..., n - 1$ all the arcs (u_n, u_i) are strong. If $\mu(u_n, u_i) = \mu(e)$, then the arcs on the cycle $e \in E(C_{n-1})$ are strong. Otherwise, they are δ-arcs. For each $e \in E(C_{n-1})$ the value of $\mu(e)$ is different, so the single fuzzy cut node is u_n. $s(u_i) = h(\mu)$ since u_n is adjacent to all nodes u_i for $i = 1, ..., n$.
Let S be a fuzzy node cut of W_n.
For $|S| = 1$, $s(S) = h(\mu)$ where $S = \{u_n\}$.
For $|S| \geq 2$, $s(S) \geq 2h(\mu)$. Thus

$$\kappa(G) = \min\{s(S)\} = h(\mu).$$

□

Theorem 6. *Let the fuzzy wheel graph W_n consist of nodes u_n as the fuzzy hub and nodes $u_1, u_2, ..., u_{n-1}$ on the fuzzy cycle C_{n-1}. If for $i = 1, ..., n - 1$, $\mu(u_n, u_i)$ values are constant and for $e \in E(C_{n-1})$, $\mu(u_n, u_i) < \mu(e)$ then the fuzzy node connectivity value is*

$$\kappa(W_n) = \begin{cases} d(\mu), & \text{if there exist FCN;} \\ 2d(\mu), & \text{otherwise.} \end{cases}$$

Proof. Let assume W_n be a fuzzy wheel graph and the value of $\mu(u_n, u_i)$ be constant for $i = 1, ..., n-1$ and $\mu(u_n, u_i) < \mu(e)$ for $e \in E(C_{n-1})$. On the fuzzy wheel graph W_n, every path is a strong path. Therefore each arc of W_n is either α-strong or β-strong. All the arcs (u_n, u_i) are β-strong and $\mu(u_n, u_i) = d(\mu)$, since u_n is adjacent to all u_i nodes and $\mu(u_n, u_i) < \mu(e)$ for all $e \in E(C_{n-1})$. Thus u_n, the fuzzy hub, is not a fuzzy cut node. Hence for $i = 1, ..., n$, $s(u_i) = d(\mu)$ and $s(S) = |S|d(\mu)$ where S represents a fuzzy node cut of W_n.

Case i) Let consider a fuzzy wheel graph W_n which contains a fuzzy cut node, represented by x and let $S = \{x\}$ be a fuzzy node cut of W_n. If x is a common vertex of at least two fuzzy bridges, then it is a fuzzy cut node. Also, if and only if the arc is α-strong, an arc is a fuzzy bridge. Therefore, it can be concluded that x is incident to the α-strong arcs (y, x) and (x, z) where $y, z \in \sigma^*(W_n)$.
Let $|S| = 1$ for $S = \{x\}$, since

$$CONN_{G-S}(y, z) = d(\mu) = q_1 < CONN_G(y, z) = q_i, i = 2, ..., m$$

then $s(S) = |S|d(\mu) = d(\mu)$.
Let $|S| \geq 2$, then $s(S) = |S|d(\mu) \geq 2d(\mu)$. By the definition we get the result

$$\kappa(G) = d(\mu).$$

Case ii) Let W_n do not have a fuzzy cut node. Therefore $|S| \geq 2$. Thus $s(S) = |S|d(\mu) \geq 2d(\mu)$. In the case of an arc (u, v) of the fuzzy wheel W_n that is not strong, it can be stated that each path between the nodes u and v is the strongest path, given that every path has the same strength between u and v, which lie on the fuzzy cycle C_{n-1}. According to Menger's theorem, the number of internally disjoint strongest pathways between nonadjacent nodes in a fuzzy cycle C_{n-1} is at most two (Mathew & Sunitha, 2013). Therefore, the cardinality of the smallest set that minimizes the strength of connectivity between these nodes is at least two.
Therefore if $|S| = 2$, then we get $s(S) = |S| = d(\mu) \Rightarrow s(S) = 2d(\mu)$. Hence

$$\kappa(G) = 2d(\mu).$$

□

Theorem 7. *Let $G : (V, \sigma, \mu)$ be a fuzzy star graph. The fuzzy node connectivity value of G is*

$$\kappa(G) = d(\mu).$$

Proof. Let G be a fuzzy star graph with n nodes where v represents the node of degree $n - 1$ and u_i represents the nodes of degree one for $i = 1, ..., n$. Since the arcs (v, u_i) are $\alpha - strong$ arcs, all the paths between two nodes are strong. Besides the node v is adjacent to all u_i for $i = 1, ..., n$, v is a fuzzy cut node. Let S be a fuzzy cut set of G.
If $|S| = 1$, then $S = \{v\}$ and $s(v) = \min\{q_i \mid i = 1, ..., m\} = q_1 = d(\mu)$.
If $|S| \geq 2$, then $s(S) = \sum_{i=1} s(u_i) \geq \sum_{i=1} q_i$, $q_i \in \mu(e)$, $e \in E(G)$. Thus the fuzzy node connectivity value is

$$\kappa(G) = d(\mu).$$

□

4 Algorithms

4.1 Algorithm 1

An algorithm to find the strength of connectedness between any two nodes in a fuzzy graph $G : (V, \sigma, \mu)$ is defined by Altundag in (Altundag, 2021) given below.

- Step 1. Write the adjacency matrix of the fuzzy graph G as matrix A.
- Step 2. Obtain the matrix AA by replacing the diagonal elements with ∞ of the matrix A.
- Step 3. If $\min\{AA[i,k], AA[k,j] > AA[i,j]$, then write $C[i,j] = \min\{AA[i,k], AA[k,j]\}$, otherwise $C[i,j] = AA[i,j]$ as the matrix of strength of connectedness.

If this algorithm applies to a fuzzy graph G, then the results are obtained as:

1. The strength of connectedness between each nodes of G are obtained.
2. Determines the strong arcs where $AA[i,j] \neq 0$. If $AA[i,j] < C[i,j]$, then (i,j) is a $\delta-$arc and if $AA[i,j] = C[i,j]$, then (i,j) is an $\alpha-$strong arc or a $\beta-$strong arc.
3. The strong weights of each nodes are obtained by $s(i) = \min\{AA[i,j] | AA[i,j] = C[i,j]\}$

Example 1. *Let $G : (V, \sigma, \mu)$ be a fuzzy graph where $\sigma^* = \{a, b, c, d\}$ with $\mu(a,b) = 0.1$, $\mu(b,c) = 0.4$, $\mu(c,d) = 0.3$, $\mu(d,a) = 0.2$ and $\mu(a,c) = 0.2$. When the algorithm is applied the matrix AA, the adjacency matrix with ∞ on the diagonals, and the strength of connectedness matrix C are obtained.*

$$AA[i,j] = \begin{bmatrix} \infty & 0.1 & 0.2 & 0.2 \\ 0.1 & \infty & 0.4 & 0 \\ 0.2 & 0.4 & \infty & 0.3 \\ 0.2 & 0 & 0.3 & \infty \end{bmatrix}$$

$$C[i,j] = \begin{bmatrix} \infty & 0.2 & 0.2 & 0.2 \\ 0.2 & \infty & 0.4 & 0.3 \\ 0.2 & 0.4 & \infty & 0.3 \\ 0.2 & 0.3 & 0.3 & \infty \end{bmatrix}$$

The results are obtained according to the above matrices are given in detailed.

1) *The strength of connectedness value between each nodes of G:*
$CONN_G(a,b) = 0.2$, $CONN_G(a,c) = 0.2$, $CONN_G(a,d) = 0.2$,
$CONN_G(b,c) = 0.4$, $CONN_G(b,d) = 0.3$, $CONN_G(c,d) = 0.3$.

2) *Strong arcs of G:*
$AA[a,b] = 0.1 < C[a,b] = 0.2 \Rightarrow (a,b)$ *δ-arc,*
$AA[a,c] = 0.2 = C[a,c] = 0.2 \Rightarrow (a,c)$ *strong arc,*
$AA[a,d] = 0.2 = C[a,d] = 0.2 \Rightarrow (a,d)$ *strong arc,*
$AA[b,c] = 0.4 = C[b,c] = 0.4 \Rightarrow (b,c)$ *strong arc,*
$AA[c,d] = 0.3 = C[c,d] = 0.3 \Rightarrow (c,d)$ *strong arc.*

3) *The strong weights of each nodes of G:*
$s(a) = \min\{(a,c),(a,d)\} = \min\{0.2, 0.2\} = 0.2$,
$s(b) = \min\{(b,c)\} = \min\{0.4\} = 0.4$,
$s(c) = \min\{(c,a),(c,b),(c,d)\} = \min\{0.2, 0.4, 0.3\} = 0.2$,
$s(d) = \min\{(d,a),(d,c)\} = \min\{0.2, 0.3\} = 0.2$.

4.2 Algorithm 2

An algorithm to find the node connectivity of a fuzzy graph $G : (V, \sigma, \mu)$ is defined by Altundag in (Altundag, 2021). This algorithm uses the Algorithm 1 in 4.1 and assumes $CONN_G(i,j) = p$ and $v \in \sigma^*$.

- Step 1. Determine the strong weights of each nodes of G by using the Algorithm 1. $s(i) = \min\{AA[i,j] | AA[i,j] = C[i,j]\}$

- Step 2. Find a set $S \subseteq \sigma^*$ which is a fuzzy node cut for G.

- Step 3. Evaluate the value of $s(S)$ using the strong weights found in Step 1 of the nodes in S.

- Step 4. Find the adjacency matrix of the graph $G - S$ by replacing the entries on the columns and the rows, corresponding to the nodes in S, of the adjacency matrix G with zero.

- Step 5. Obtain the matrix of strength of connectedness C for the fuzzy graph $G - S$ by using the Algorithm 1. For the Step 1 of the Algorithm 1 use the adjacency matrix obtained in Step 4 above.

- Step 6. If $q < p$, then $K = s(S)$ where $C[i,j] = p$ for the fuzzy graph G and $C[i,j] = q$ for the fuzzy graph $G - S$, otherwise go to Step 2.

- Step 7. Repeat the Steps 2 through 6 until there is no other set S exists in Step 2.

- Step 8. Find the minimum of all the values of K which gives the node connectivity value $\kappa(G)$ of the fuzzy graph G.

5 Conclusion

Fuzzy graphs are a special type of graphs. Because in real-life problems modelling by grading the relations will be more closer to reality. In situations where there is ambiguity surrounding the nodes and/or links, fuzzy graphs emerge as a particularly useful tool. Therefore, fuzzy graphs represent all systems more accurately than classical graphs, depending on the uncertainty or fuzziness of the system parameters. Nevertheless, there is a scarcity of research examining the vulnerability of fuzzy graphs. Considering the usage areas of fuzzy graphs, this is an important deficiency. In this paper, the node connectivity of some types of fuzzy graphs are studied. The general formulas have been derived for fuzzy wheel graphs, fuzzy cycle graphs, and fuzzy star graphs.Additionally, an algorithm is provided for determining the strength of connectedness between any two nodes, as well as another algorithm for evaluating the node connectivity of the given fuzzy graph.

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